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UNCERTAINTIES IN TRACK MOMENTUM AND DIRECTION, DUE TO MULTIPLE SCATTERING AND MEASUREMENT ERRORS*

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The effect of measurement errors and multiple scattering on the uncertainties in momenta and direction of bubble chamber tracks is reexamined. Results are given for the rms uncertainty in momentum and direction, and their correlation, for uniform weighting, and both uniform spacing and clustering of the measured points. It is shown that for measurement errors, clustering of measured points at the beginning, end, and center of the track leads to lower rms uncertainties. Estimates of the effect of multiple scattering include the contribution of single and plural small angle scattering outside the central Gaussian region. The contribution of atomic electrons to the multiple scattering is treated separately. In addition the effect of small angle nuclear scattering is included. Some numerical results are presented for a variety of projectiles in liquid hydrogen and propane chambers.

1. Introduction

In the reconstruction of particle directions and momenta from the measurement of track coordinates, uncertainties in the results (curvature and angle) arise from many sources. Among these, the most important usually are the limitation in the accuracy with which track coordinates can be measured[1]) and the error introduced by the multiple scattering of the particle in question[2]).

In the application of multiple scattering theory to the problem of obtaining particle momenta and directions, there has been a tendency recently to treat only the central Gaussian region of the multiple scattering distribution. Of course this neglects the effect of single and plural scattering which also contributes to the uncertainty in direction and momentum. A similar omission has occurred in the treatment of the correction to Mott scattering due to the finite thickness of scattering foils[3]). Inclusion of single and plural scattering[4]) in this case has even led to a different form of the dependence of these corrections on scatterer thickness.

The present work reviews the effect of multiple scattering on track reconstruction, including the effect of single and plural small angle scattering. For completeness the subject of measurement errors is also included. In sec. 2 general formulas are given for the effect of arbitrary coordinate errors on the calculated curvature and direction. These formulas are applied to the case of measurement error, where both optimum weighting and spacing of the measured points is discussed. Formulas are also given for the effect of multiple scattering which depends only on the rms angle of multiple scattering and not on the distribution itself. In sec. 3 the appropriate formulas for the rms angle per unit scatterer thickness are derived and applied to a variety of projectiles and scatterers. Some numerical results are given to aid in the computation of the rms angle. Sec. 4 contains a discussion of the results of this paper.

2. Errors in Curvature and Direction

Consider the measurement of the coordinates of a track projected onto a plane perpendicular to the assumed uniform field. If one neglects the energy variation along the track, it can be shown that the curvature and direction errors are the same as those which would exist in the absence of the magnetic

* Supported in part by the Atomic Energy Commission.

1) Eisler, Gregory, Rau, Thorndike and Willis, Internal Brookhaven memoranda.

2) W. T. Scott, Phys. Rev. **76** (1949) 212;
H. A. Bethe, Phys. Rev. **70** (1946) 821.

3) H. Wegener, Z. Phys. **151** (1958) 252.

4) Greenberg, Malone, Gluckstern and Hughes, Phys. Rev. **120** (1960) 1393.

field. For a track of projected length L with transverse (to the magnetic field direction) coordinates y_n measured at longitudinal coordinates x_n ($n = 0, 1 \ldots, N$), the curvature and direction errors are the values of c and θ in the least square fit of

$$y = \alpha + \theta x + \tfrac{1}{2}cx^2 \tag{1}$$

to the measured coordinates. If the weighting of each point is f_n, one obtains the set of linear equations

$$\begin{aligned} \alpha F_0 + \theta F_1 + \tfrac{1}{2}c\,F_2 &= \sum y_n f_n\,, \\ \alpha F_1 + \theta F_2 + \tfrac{1}{2}c\,F_3 &= \sum y_n f_n x_n\,, \\ \alpha F_2 + \theta F_3 + \tfrac{1}{2}c\,F_4 &= \sum y_n f_n x_n^2\,, \end{aligned} \tag{2}$$

where

$$F_j = \sum_0^N f_n (x_n)^j\,. \tag{2a}$$

This leads to*

$$\tfrac{1}{2}c = \sum y_n Q_n \Big/ \sum x_n^2 Q_n\,, \quad \theta = \sum y_n P_n \Big/ \sum x_n P_n\,, \tag{3}$$

where

$$\begin{aligned} Q_n &= f_n \left\{ \begin{vmatrix} F_1 F_2 \\ F_2 F_3 \end{vmatrix} - x_n \begin{vmatrix} F_0 F_1 \\ F_2 F_3 \end{vmatrix} + x_n^2 \begin{vmatrix} F_0 F_1 \\ F_1 F_2 \end{vmatrix} \right\} \\ P_n &= f_n \left\{ - \begin{vmatrix} F_1 F_3 \\ F_2 F_4 \end{vmatrix} + x_n \begin{vmatrix} F_0 F_2 \\ F_2 F_4 \end{vmatrix} - x_n^2 \begin{vmatrix} F_0 F_1 \\ F_2 F_3 \end{vmatrix} \right\}. \end{aligned} \tag{4}$$

One can readily show that

$$\sum Q_n = \sum x_n Q_n = \sum P_n = \sum x_n^2 P_n = 0\,, \tag{5}$$

and

$$\sum x_n^2 Q_n = \sum x_n P_n\,. \tag{6}$$

The r.m.s. errors and correlation are therefore given by

$$\begin{aligned} \langle c^2 \rangle &= 4 \sum Q_m Q_n \langle y_n y_m \rangle \Big/ (\sum x_n^2 Q_n)^2 \\ \langle c\theta \rangle &= 2 \sum Q_m P_n \langle y_n y_m \rangle \Big/ (\sum x_n^2 Q_n)(\sum x_n P_n) \\ \langle \theta^2 \rangle &= \sum P_m P_n \langle y_n y_m \rangle \Big/ (\sum x_n P_n)^2\,. \end{aligned} \tag{7}$$

These expressions are clearly independent of the normalizations of P_n and Q_n. In what follows the normalizations will be adjusted for convenience.

* All sums will be understood to run from $n = 0$ to $n = N$.

2.1. MEASUREMENT ERRORS

For uncorrelated measurement errors of r.m.s. magnitude ε one has

$$\langle y_n y_m \rangle = \varepsilon^2 \delta_{mn} \tag{8}$$

leading to

$$\begin{aligned} \langle c^2 \rangle &= \varepsilon^2\, 4 \sum Q_n^2 \Big/ (\sum x_n^2 Q_n)^2 \\ \langle c\theta \rangle &= \varepsilon^2\, 2 \sum Q_n P_n \Big/ (\sum x_n^2 Q_n)(\sum x_n P_n) \\ \langle \theta^2 \rangle &= \varepsilon^2 \sum P_n^2 \Big/ (\sum x_n P_n)^2\,. \end{aligned} \tag{9}$$

2.1.1. *Weightings*

The particular assignment of weightings, f_n, which minimizes these errors can be determined by minimizing (9) with arbitrary P_n, Q_n subject only to the requirements (5) and (6). For example, the smallest $\langle c^2 \rangle$ is obtained with that set of Q_n which minimizes $\sum Q_n^2$ subject to

$$\sum Q_n = 0\,, \quad \sum x_n Q_n = 0\,, \quad \sum x_n^2 Q_n = L^2\,, \tag{10}$$

where the last condition represents a convenient normalization for the Q_n. The method of Lagrange multipliers leads immediately to the solution

$$Q_n = a - bx_n + dx_n^2 \tag{11}$$

where a, b, d are determined such that (10) is satisfied. Comparison with (4) indicates that the optimum weighting is *uniform*, a condition which also applies for the minimum $\langle c\theta \rangle$ and $\langle \theta^2 \rangle$.

2.1.2. *Uniform Spacing*

Apart from normalization, the values for Q_n and P_n are given by

$$\begin{aligned} Q_n &= n^2 - nN + \tfrac{1}{6}N(N-1) \\ P_n &= -15Nn^2 + (2N+1)(8N-3)n \\ &\quad - \tfrac{3}{2}N(N-1)(2N+1) \end{aligned} \tag{12}$$

The sums indicated in (2a) and (9) can be performed for uniform spacing, leading to*

* The relation $\langle c^2 \rangle = -\langle 2c\theta \rangle$ implies no correlation between the direction at the center of the track and the curvature.

$$\langle c^2 \rangle = \frac{\varepsilon^2}{L^4} \cdot \frac{720N^3}{(N-1)(N+1)(N+2)(N+3)} = \frac{\varepsilon^2}{L^4} A_N$$

$$\langle c\theta \rangle = -\frac{\varepsilon^2}{L^3} \cdot \frac{360N^3}{(N-1)(N+1)(N+2)(N+3)} = -\tfrac{1}{2}\frac{\varepsilon^2}{L^3} A_N \qquad (13)$$

$$\langle \theta^2 \rangle = \frac{\varepsilon^2}{L^2} \cdot \frac{12(2N+1)(8N-3)N}{(N-1)(N+1)(N+2)(N+3)} = \frac{\varepsilon^2}{L^2} B_N .$$

Table 1 contains the values of A_N, B_N for various values of N for the case of uniform spacing of measured points. The parameters A'_N and B'_N are discussed in the following section.

TABLE 1
Parameters for r.m.s. curvature and direction uncertainties due to measurement errors

N	A_N	B_N	A'_N	B'_N
2	96	26	—	—
3	81	22.1	64	18
4	73.1	19.9	—	—
5	67.0	18.2	—	—
6	61.7	16.7	—	—
7	57.2	15.5	32	9
8	53.2	14.4	—	—
∞	$\frac{720}{N+5}$	$\frac{192}{N+4.9}$	$\frac{256}{N+1}$	$\frac{72}{N+1}$

2.1.3. *Optimum Spacing*

If one considers the total number of measured points $(N+1)$ and the projected track length $L(x_0 = 0, x_N = L)$ fixed, one can ask what spacing of the internal points will lead to the minimum $\langle c^2 \rangle$, $\langle c\theta \rangle$, $\langle \theta^2 \rangle$. The optimum spacing is *different* in each case and will be derived here for the minimum $\langle c^2 \rangle$. One wishes to minimize $\sum Q_n^2$ subject to the conditions (10), with $x_1, x_2 \ldots x_{N-1}$ as variables. For a change δx_n in x_n one wishes to have, using (12)

$$\sum Q_n \delta Q_n = a \sum \delta Q_n - b \sum x_n \delta Q_n + c \sum x_n^2 \delta Q_n = 0 \quad (14)$$

subject to

$$\sum \delta Q_n = 0, \quad \sum x_n \delta Q_n = -\sum Q_n \delta x_n, \quad \sum x_n^2 \delta Q_n = -2 \sum Q_n x_n \delta x_n \qquad (14a)$$

from (10). This leads to

$$\sum Q_n \delta x_n (b - 2d x_n) = 0 \qquad (15)$$

which implies that all *internal* points should have the same track coordinate $x_n = b/2d$. It can readily be shown that the minimum value of $\langle c^2 \rangle$ can be obtained with a cluster of $\frac{1}{4}(N+1)$ *independent* points taken at $x = 0$, $\frac{1}{2}(N+1)$ points at $x = \frac{1}{2}L$ and $\frac{1}{4}(N+1)$ points at $x = L$. This prescription leads to the values*

$$\langle c^2 \rangle = \frac{\varepsilon^2}{L^4} \frac{256}{N+1} = \frac{\varepsilon^2}{L^4} A'_N$$

$$\langle c\theta \rangle = -\frac{\varepsilon^2}{L^3} \frac{128}{N+1} = -\tfrac{1}{2}\frac{\varepsilon^2}{L^3} A'_N \qquad (16)$$

$$\langle \theta^2 \rangle = \frac{\varepsilon^2}{L^2} \frac{72}{N+1} = \frac{\varepsilon^2}{L^2} B'_N .$$

For comparison with A_N and B_N one finds (see table 1)

$$A'_3 = 64, \quad B'_3 = 18, \quad A'_7 = 32, \quad B'_7 = 9 . \qquad (17)$$

2.2. MULTIPLE SCATTERING ERRORS

For multiple scattering errors, one starts with (7), using the appropriate value for the average correlation $\langle y_n y_m \rangle$. It can be shown[5]) from general considerations independent of the actual distribution of displacements and angles that for a scatterer of thickness t, the r.m.s. projected angle, displacement and correlation are given by

* The optimum for $\langle c\theta \rangle$ and $\langle \theta^2 \rangle$ is slightly lower than that given in (16). For example, $\langle \theta^2 \rangle_{\text{opt}} = 64/N+1$ for $\frac{3}{8}(N+1)$, $\frac{1}{2}(N+1)$, $\frac{1}{8}(N+1)$ points taken at $x = 0$, $\frac{1}{2}L$ and L respectively.

[5]) E.g.: E. J. Williams, Proc. Roy. Soc. A**168** (1939) 531; Rossi and Greisen, Rev. Mod. Phys. **13** (1941) 240; also ref.[4]).

$$\langle \psi_y^2 \rangle = Kt$$
$$\langle y\psi_y \rangle = \tfrac{1}{2}Kt^2 \qquad (18)$$
$$\langle y^2 \rangle = \tfrac{1}{3}Kt^3$$

where K is the r.m.s. projected angle per unit thickness for multiple scattering and is discussed in detail in sec. 3. Using (18) one finds

$$\langle y_n y_m \rangle = \tfrac{1}{2}Kx_n^2(3x_m - x_n), \; x_m \geq x_n \qquad (19)$$

with $\langle c^2 \rangle$, $\langle c\theta \rangle$ and $\langle \theta^2 \rangle$ now given by (7) and (19).

The question of the optimum weighting and spacing to use for the curvature due to multiple scattering has been considered by Scott[2]). He starts with the general expression for the curvature for uniform spacing:

$$c = \sum_1^{N-1} q_n(y_{n+1} - 2y_n + y_{n-1}) , \qquad (20)$$

and is led to an optimum choice of weights q_n which minimize $\langle c^2 \rangle$. In the present language, these correspond to weights f_n, some of which are negative, but are not otherwise objectionable. However these optimum weights lead to a curvature due to measurement errors which is over a factor of 2 greater than the errors given in (16). For all but the lowest energy particles the measurement errors are most serious and a selection of weights according to either the uniform or optimum spacing is desirable[6]).

2.2.1. *Uniform Spacing*

For uniform spacing one finds

$$\langle c^2 \rangle_{\text{M.S.}} = \frac{K}{L} C_N$$
$$\langle c\theta \rangle_{\text{M.S.}} = -KD_N$$
$$\langle \theta^2 \rangle_{\text{M.S.}} = KL\,E_N \qquad (21)$$

The expressions for C_N, D_N, E_N are quite complicated but can be constructed from (7) using (12) and (19). Values are given in table 2.

[6]) Optimum weights for arbitrary ratio of multiple scattering to measurement errors have been given by M. Huybrechts, Bull. Acad. Roy. Belg. Cl. Sc. **6** (1961) 515, 739.

The author would like to thank F. Solmitz for bringing these references to his attention.

TABLE 2
Parameters for r.m.s. curvature and direction uncertainties due to multiple scattering

N	C_N	D_N	E_N
2	1.33	0.167	0.167
3	1.25	0.125	0.154
4	1.25	0.124	0.160
5	1.26	0.132	0.167
9	1.31	0.156	0.187
∞	1.43	0.214	0.229

It should be mentioned that Scott's optimum choice of weights leads to a $C'_\infty = 1.00$.

2.2.2. *Optimum Spacing (for minimum curvature error due to measurement errors)*

The measurement of clustered points at $x = 0$, $\frac{1}{2}L$, L leads to errors due to multiple scattering identical with the values listed above for $N = 2$. The reason is that a cluster of points is equivalent to a single point as far as multiple scattering is concerned. One therefore has

$$\langle c^2 \rangle_{\text{M.S.}}^{\text{opt.}} = \frac{K}{L} \cdot \tfrac{4}{3}$$
$$\langle c\theta \rangle_{\text{M.S.}}^{\text{opt.}} = -K \cdot \tfrac{1}{6} \qquad (22)$$
$$\langle \theta^2 \rangle_{\text{M.S.}}^{\text{opt.}} = KL \cdot \tfrac{1}{6}$$

2.3. R.M.S. FITS TO A STRAIGHT LINE

For completeness, we list the appropriate formulas for the r.m.s. error in direction when fitting to a straight line (appropriate along the magnetic field direction). Here one has

$$\langle \theta_{\text{SL}}^2 \rangle = \sum \langle y_n y_m \rangle R_n R_m \Big/ (\sum x_n R_n)^2 \qquad (23)$$

where

$$R_n = F_1 - x_n F_0 . \qquad (24)$$

2.3.1. *Measurement Errors*

For uniform spacing one obtains the result

$$\langle \theta_{\text{SL}}^2 \rangle = \frac{\varepsilon^2}{L^2} \frac{12N}{(N+1)(N+2)} \qquad (25)$$

where L is now the rectified track length.

For a clustering of $\frac{1}{4}(N+1)$, $\frac{1}{2}(N+1)$, $\frac{1}{4}(N+1)$ points at $x = 0$, $\frac{1}{2}L$, L, one finds

$$\langle\theta_{SL}^2\rangle = \frac{\varepsilon^2}{L^2}\frac{8}{N+1}. \quad (26)$$

The optimum $\langle\theta_{SL}^2\rangle$ is clearly obtained with a cluster of $\frac{1}{2}(N+1)$, $\frac{1}{2}(N+1)$ points at $x = 0, L$. It is

$$\langle\theta_{SL}^2\rangle = \frac{\varepsilon^2}{L^2}\frac{2}{N+1}. \quad (27)$$

2.3.2. *Multiple Scattering Errors*

The r.m.s. error in direction due to multiple scattering turns out to be the same for a straight line fit to two points (end clusters) or three equally spaced points (clusters at 0, $\frac{1}{2}L$, L). It is

$$\langle\theta_{SL}^2\rangle = \tfrac{1}{3}KL. \quad (28)$$

3. Mean Square Angle Due to Multiple Scattering

It was recognized long ago[5]) that the r.m.s. angle and displacement due to multiple scattering could be given without knowing the actual distributions. The first efforts indicated that the distribution was Gaussian, but this was recognized to be approximate, and a great deal of effort was spent deriving the correct distributions. Among those who have obtained a useful approximation to the distribution is Moliere[7]).

In the application of multiple scattering theory as a correction to the observations in a variety of experiments, the fact that both the central Gaussian region as well as the plural tail should be taken into account has frequently been overlooked. In particular, those authors who have used the 1/e point of Moliere's distribution as the r.m.s. angle have neglected contributions from small angle single and plural scattering. This has been frequently true in the expressions used by many people for the parameter K in (18).

Let us start with the Boltzmann equation for the distribution in direction $\boldsymbol{n}$, at a depth z in the scatterer:

$$\frac{\partial f(\boldsymbol{n}, z)}{\partial z} = -Nf(\boldsymbol{n}, z)\int\frac{\sigma(\boldsymbol{n}\to\boldsymbol{n}')}{\boldsymbol{n}\cdot\boldsymbol{n}_0}\,d\Omega' + N\int f(\boldsymbol{n}', z)\frac{\sigma(\boldsymbol{n}'\to\boldsymbol{n})}{\boldsymbol{n}'\cdot\boldsymbol{n}_0}\,d\Omega' \quad (29)$$

where $\boldsymbol{n}_0$ is a unit vector in the z direction, N is the number of scatterers per unit volume and $\sigma(\boldsymbol{n}_1\to\boldsymbol{n}_2)$ is the cross section for scattering from direction $\boldsymbol{n}_1$ to direction $\boldsymbol{n}_2$. If ψ is the angle between $\boldsymbol{n}$ and $\boldsymbol{n}_0$, it can be shown for an azimuthally symmetric scattering law, that

$$\langle 1-\cos\psi\rangle = Nz\int(1-\cos\theta)\sigma(\theta)\,d\Omega. \quad (30)$$

In deriving (30), it is assumed that there is no backward flux. The parameter K, representing the r.m.s. projected angle per unit thickness, is therefore given approximately by

$$K = N\pi\int\theta^3\sigma(\theta)\,d\theta. \quad (31)$$

For a Rutherford scattering law,

$$\sigma(\theta) = 4(Ze^2/p\beta c)^2\,\theta^{-4}, \quad (32)$$

one has

$$K = 4\pi N(Ze^2/p\beta c)^2\ln(\theta_{max}/\theta_{min}). \quad (33)$$

The angle θ_{min} is determined by atomic screening and θ_{max} is determined by the geometry of the particular application. A common procedure is to replace Z^2 in (33) by $Z(Z+1)$ to take into account scattering from the atomic electrons.

3.1. SCREENING ANGLE

The angle θ_{min} is usually taken for high Z atoms from a Fermi-Thomas description of the electronic charge density. A modification due to Moliere leads to[7])

$$\theta_{min}^z \simeq \left\{1.167\left[1.13+3.76\left(\frac{z}{137\beta}\right)^2\right]\right\}^{\frac{1}{2}}\left(\frac{\lambda Z^{\frac{1}{3}}}{0.885a_0}\right) \quad (34)$$

For atomic hydrogen one can use the actual scattering law appropriate to the ground state to obtain

$$\theta_{min}^1 \simeq 1.116\frac{\lambda}{a_0} = \frac{4.2\times10^{-3}}{(p)_{MeV/c}}. \quad (35)$$

Modifications due to molecular binding should change θ_{min}^1 by no more than about 20%.

[7]) G. Moliere, Z. Naturforsch. **3a**, (1948) 78;
H. A. Bethe, Phys. Rev. **89** (1953) 1256;
also B. P. Nigam, M. K. Sundaresan and T. Y. Wu, Phys. Rev. **115** (1959) 491.

3.2. MAXIMUM SCATTERING ANGLE

If the angle θ_{max} is chosen corresponding to the 1/e point of the Gaussian distribution, the single and plural scattering tails will *not* have been taken into account. Inclusion of these tails, which is appropriate for the assignment of curvature and direction errors, requires choosing θ_{max} to be appropriate either to contact with the nucleus or to an observable kink in the track. Thus one must choose θ_{max} to be the smaller of θ_K, the minimum observable projected kink angle for the track, or $\lambda\bar{}/r_N$, i.e.

$$\theta_{max} \sim \begin{cases} \theta_K & \lambda\bar{} > r_N\theta_K \\ \lambda\bar{}/r_N\,, & \lambda\bar{} < r_N\theta_K\,, \end{cases} \tag{36}$$

where r_N is the nuclear radius.

The above is not quite correct, since the prescription for selecting θ_{max} is necessarily different for different incident particles. For example, (36) should be approximately valid for the multiple scattering of high energy electrons or muons if r_N is chosen to be appropriate to the r.m.s. radius for the nuclear charge distribution, as described by the experimental form factor. For scattering of strongly interacting particles, however, one must augment the Coulomb scattering by small angle (0°) nuclear scattering out to an angle θ_K. For the present estimates one can neglect the Coulomb-nuclear interference. Using the optical theorem for forward scattering, one finds that (33) is to be replaced by

$$K = K_1 + K_3\,, \tag{37}$$

where

$$K_1 = 4\pi N \left(\frac{Ze^2}{p\beta c}\right)^2 \ln\left(\frac{\theta_K}{\theta_{min}}\right) \tag{38}$$

$$K_3 = \frac{\pi N}{4}\left(\frac{p\sigma_{tot}}{4\pi\hbar}\right)^2 \theta_K^4\,. \tag{39}$$

Here σ_{TOT} is the appropriate total nuclear cross section. For nuclear cross sections in the range 20–60 mb it is found that K_1 and K_3 are approximately equal for momenta of the order of 3–10 GeV/c.

3.3. SCATTERING FROM ATOMIC ELECTRONS

It is necessary, particularly for low Z, to include scattering from the atomic electrons in the calculation of K. The usual prescription to replace Z^2 in (33) by $Z^2 + Z$ is a procedure which can be justified only if $\theta_{max}/\theta_{min}$ remains the same. Although the estimate of θ_{min} in (34) or (35) should be of the right order, the maximum angle is reduced from θ_K because of the inability of an electron to deflect a heavy particle. In fact one finds

$$\theta^e_{max} \sim 1.2m_e/\sqrt{M^2 + 2m_eE} \tag{40}$$

where M and E are the mass and energy of the incident particle. The expression $\theta^e_{max} = 1.2\ m_e/M$ is clearly valid for presently available accelerator energies for all except incident electrons, for which $\theta^{ee}_{max} \sim 1.2\ (m_e/E_e)^{\frac{1}{2}}$. The contribution to K due to scattering from atomic electrons can therefore be written as

$$K_2 = 4\pi NZ\left(\frac{e^2}{p\beta c}\right)^2 \ln\left(\frac{\theta^e_{max}}{\theta_{min}}\right)\,. \tag{41}$$

3.4. SUMMARY OF MULTIPLE SCATTERING FORMULAS

The total contribution is

$$K = K_1 + K_2 + K_3 \tag{42}$$

where K_1 is given by (38) or (33) and (36), K_2 is given by (41) and K_3 is given by (39). Listed below are approximate values for K_1, K_2, K_3 for liquid hydrogen ($N = 3.4 \times 10^{22}$ atoms/cm^3) bubble chambers, with θ_K taken as 0.02 radians in each case.

3.4.1. *Liquid Hydrogen*

a. $$K_1 = \frac{0.0098}{\beta^2(p_{MeV/c})^2} S_1\ \text{cm}^{-1}\,. \tag{43}$$

For electrons*, muons*,

$$S_1 = \begin{cases} \ln(4.8p_{MeV/c})\,, & p < 10\ \text{GeV}/c \\ 10.8\,, & p > 10\ \text{GeV}/c \end{cases}\,. \tag{44}$$

For strongly interacting particles

* The electron (and muon) scattering cross sections should be modified to take into account collisions in which radiation is emitted. This leads to a reduction of S_1 and S_2 of order 1–3% ($\mathcal{O}(1/137)$) which has been neglected here.

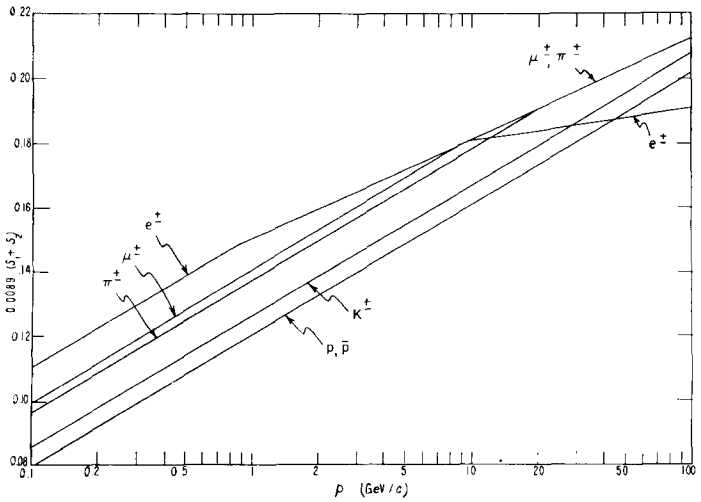

Fig. 1. Plot of the parameter entering into the mean square angle for multiple scattering in liquid hydrogen as a function of momentum.

$$S_1 = \ln (4.8 p_{\mathrm{MeV}/c}) \,. \tag{45}$$

b.
$$K_2 = \frac{0.0089}{\beta^2 (p_{\mathrm{MeV}/c})^2} S_2 \ \mathrm{cm}^{-1} \tag{46}$$

For electrons

$$S_2 = \left\{ \begin{array}{l} \ln (4.8 p_{\mathrm{MeV}/c}) \ , \ p < 900 \ \mathrm{MeV}/c \\ \\ \ln (145 p^{\pm}_{\mathrm{MeV}/c}) \ , \ p > 900 \ \mathrm{MeV}/c \end{array} \right\} . \tag{47}$$

For mesons, nucleons, hyperons, etc.

$$S_2 = \left\{ \begin{array}{ll} \ln (145 p/Mc) & p < M^2 \ \mathrm{MeV}/c \\ \\ \ln (145 p^{\pm}_{\mathrm{MeV}/c}) \ , & p > M^2 \ \mathrm{MeV}/c \end{array} \right\} \tag{48}$$

with M in MeV/c^2.

c. For electrons, muons

$$K_3 = 0 \,. \tag{49}$$

For strongly interacting particles

$$K_3 = 7 \times 10^{-20} p^2_{\mathrm{MeV}/c} \cdot \sigma^2_{\mathrm{tot}} \,, \tag{50}$$

where σ_{TOT} is in millibarns.

d. Plots of 0.0089 $(S_1 + S_2)$ and σ_{TOT} vs. p[8]) are given in figs. 1 and 2, for a hydrogen chamber.

3.4.2. *Propane* (C_3H_8)

a.
$$K_1 = \frac{0.013 S_1 + 0.17 S_1'}{\beta^2 p^2_{\mathrm{MeV}/c}} \,. \tag{51}$$

[8]) E.g.: G. Cocconi, 1962 Intern. Conf. on High Energy Physics, p. 883;
W. F. Baker, R. L. Cool, E. W. Jenkins, T. F. Kycia, R. H. Phillips and A. L. Read, Phys. Rev. **129** (1963) 2285.

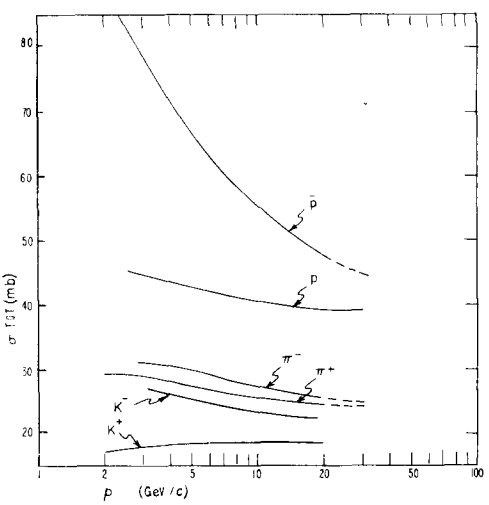

Fig. 2. Total nuclear cross section for various particles incident on protons as a function of momentum.

For electrons, muons,

$$S_1' = \left\{ \begin{array}{ll} \ln (2.3p_{\mathrm{MeV}/c}), & p < 3.5 \text{ GeV}/c \\ 9.0 & p > 3.5 \text{ GeV}/c. \end{array} \right\} . \quad (52)$$

For strongly interacting particles

$$S_1' = \ln (2.3p_{\mathrm{MeV}/c}) . \quad (53)$$

b. $$K_2' = \frac{0.013S_2 + 0.028S_2'}{\beta^2 p^2_{\mathrm{MeV}/c}} . \quad (54)$$

For all particles

$$S_2' = S_2 - 0.75 . \quad (55)$$

c. For electrons, muons

$$K_3 = 0 . \quad (56)$$

For strongly interacting particles

$$K_3 = 2.5 \times 10^{-20} p^2_{\mathrm{MeV}/c} \cdot \sigma^2_{\mathrm{tot}} , \quad (57)$$

where σ_{tot} (in mb) is again the total cross section on *protons* and the neutron and proton cross sections are assumed to be equal.

d. A plot of $0.013(S_1 + S_2) + (0.17S_1' + 0.028S_2')$ vs p is given in fig. 3.

4. Summary and Discussion

The uncertainties in curvature and direction due to measurement errors and to multiple scattering have been presented in secs. 2 and 3. In particular the following results have been obtained:

1. General formulas are given for the r.m.s. curvature and direction errors and for their correlation, for arbitrary weighting and spacing of points along the track. These are given for measurement errors and for multiple scattering errors for fits to both a circle and a straight line.

2. It is shown that the minimum uncertainties due to measurement error occur with equal weighting of the points. Explicit formulas are given for these uncertainties in the case of equal spacing of the points. Some numerical results are given.

3. It is also shown that the minimum uncertainties due to measurement error occur for clustered points. In particular, the minimum r.m.s. curvature error occurs for clusters of points at the beginning, middle and end of the track with numbers of points in each cluster being in the ratio 1:2:1. Explicit formulas are given for the uncertainties in curvature and direction for the 1:2:1 clusters. Numerical results are compared with the case of uniform spacing. For 8 points, the improvement in the uncertainties over that for uniform spacing is of the order of 30%.

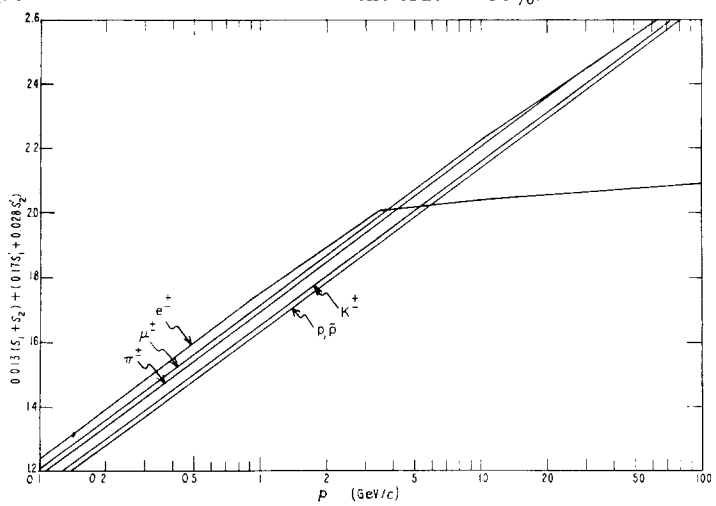

Fig. 3. Plot of the parameter entering into the mean square angle for multiple scattering in propane as a function of momentum.

4. Numerical results are presented for the uncertainties in direction and angle due to multiple scattering. These are given both for the uniform spacing and clustered cases.

5. The r.m.s. angle per unit thickness used in the multiple scattering formulas is calculated taking into account both multiple and plural small angle scattering. Among the effects taken into account are

a. atomic screening,
b. kinks in the track,
c. finite nuclear size,
d. scattering by atomic electrons,
e. small angle nuclear scattering.

Numerical estimates of the parameters which enter are included for liquid hydrogen and for propane.

It is clear that for most tracks in the GeV range, the uncertainty due to measurement error will dominate for the present state of the art. For this reason the optimum procedures for measurement error should be followed wherever possible. Nevertheless, for tracks of low momentum, or for particularly long tracks, the multiple scattering error will be important and may change the most desirable measurement procedure.

No attempt has been made to include the following effects, which may modify the optimum procedure:

a. non-linearities in the magnetic field or optics,
b. energy loss along the track,
c. turbulence in the liquid medium,
d. consistency of the stereoscopic views,
e. possible blunders in obtaining track coordinates.

The particular measurement scheme adopted will undoubtedly depend on the chamber used, the particular experiment, the computing machine available and the personal taste of the individuals involved.

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